This is an unsteady state problem, since temperature at a time is desired. If the coldest point changes temperature, it is not semi-infinite. It is thus a finite geometry. It cannot be a lumped parameter if there is discussion about temperature at different locations. Thus, we need series solution. At this stage, we cannot check if \( \frac{\alpha t}{L^2} > 0.2 \), which is needed to use one term solution. We can gamble and use one term solution and later make sure of this when we know \( t \). For one term solution, we have a choice of using the one term algebraically or read from Heisler chart. Note that for a convection boundary condition, we did not have the one term solution derived (what we derived was for temperature specified boundary condition). Thus, we have no choice but to use Heisler chart.

Given data:

\[
2L = 500\mu m \\
k = 0.5 \text{ W/m} \cdot \text{K} \\
\alpha = 1.5 \times 10^{-7} \text{ m}^2/\text{s} \\
T_i = 21^\circ \text{C} \\
h = 2 \times 10^3 \text{ W/m}^2 \cdot \text{K} \\
T_s = 65^\circ \text{C}
\]

1) Unsteady heat conduction in 1D slab:

\( T = T_c \) (the coldest point is at the center)
Use Heisler chart:

\[ m = \frac{k}{hL} = \frac{0.5}{2000 \times 250 \times 10^{-6}} = 1 \]

\[ n = \frac{x}{L} \text{ at } x = 0; \; n = 0 \]

Tissue considered isothermal when temperature at coldest point is within 2°C of fluid temperature

\[ \therefore \; \frac{T_c - T_s}{T_i - T_s} = \frac{63 - 65}{21 - 65} = 0.045 \]

\[ \text{for } \frac{T(x) - T_\infty}{T_i - T_\infty} = 0.045; \; m = 1 \text{ and } n = 0; \]

we have \( F_0 = \frac{\alpha t}{L^2} = 4.5 \) (from Heisler chart)

\[ \therefore t = \frac{(L^2)(Fo)}{\alpha} = \left( \frac{250 \mu m}{1.5 \times 10^{-7}} \right)^2 \left( 2.8 \right) = 1.875\text{s} \]

2) For both 4°C increase in initial temperature and fluid temperature, there is no change in time from (1) above.

3) For 2x the thickness,

\[ L' = 2L = 500\mu m \]

\[ m = \frac{k}{hL'} = \frac{0.5}{2000 \times 500 \times 10^{-6}} = 0.5 \]

\[ F_0 = \frac{\alpha t'}{(L')^2} = 2.8 \]

\[ t' = \frac{(L')^2(Fo)}{\alpha} = 4.68\text{s} \]
Problem 2

KNOWN:
Individual resistances

FIND:
Combined resistance

SCHEMATICAL AND GIVEN DATA

Skin
Outside surface
Body
D = 16 cm
Plumage thickness
\( t = 2 \text{ cm} \)

STRATEGY:
Resistances can be combined in series or parallel. Here, the radiative and convective resistances are in parallel while their combined resistance is in series with conductive resistance. Resistances are also in cylindrical coordinate.

ASSUMPTIONS:

SOLUTION:

1. \[ R_{\text{rad}} = \frac{\Delta T}{q_{\text{rad}}} = \frac{T - T_{\infty}}{q_{\text{rad}}} = \frac{1}{4\sigma T_{\infty}^4 A} \]

2. \[ R_{\text{conv}} = \frac{\Delta T}{q_{\text{conv}}} = \frac{T - T_{\infty}}{q_{\text{conv}}} = \frac{1}{hA} \]

3. \[ \frac{T - T_{\infty}}{R_{\text{out}}} = \frac{T - T_{\infty}}{R_{\text{conv}}} + \frac{T - T_{\infty}}{R_{\text{rad}}} \]
   \[ \frac{1}{R_{\text{out}}} = \frac{1}{R_{\text{conv}}} + \frac{1}{R_{\text{rad}}} = \frac{1}{hA + 4\sigma T_{\infty}^4 A} \]
   \[ R_{\text{out}} = \frac{1}{hA + 4\sigma T_{\infty}^4 A} \]
\[ R_{\text{cond}} = \frac{\ln \left( \frac{r_p}{r_z} \right)}{2\pi k L} \]

\[ R_{\text{total}} = \left( \ln \left( \frac{r_p}{r_z} \right) \right) + \frac{1}{\lambda A + 4\sigma T_\infty^3 A} \]

5.

\[ A = 2\pi r_p H = 2\pi \times 0.1 \times 0.43 = 0.27 \]

\[ R = \left( \frac{\ln \left(0.1/0.08\right)}{2\pi \left(0.0386\right) \left(0.43\right)} \right) + \frac{1}{70 + 4 \times 5.67 \times 10^{-8} \times (278)^3 \left(0.27\right)} = 2.19 \]

\[ q = \frac{T_z - T_\infty}{R} = \frac{34 - 5}{2.19} = 13.2 W \]

6.

\[ q_{\text{gen}} = q = 13.2 W \]

7.

GE:

For plumeage:

\[ \frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0 \]

BCs:

\[ r = r_z \]

\[ T = T_z \]

\[ r = r_p \]

\[ -k A \frac{dT}{dr} = \lambda A \left( T_{\text{plum.surface}} - T_\infty \right) + 4\sigma T_\infty^3 A \left( T_{plum.surface} - T_\infty \right) \]

\[ q = -k A \frac{dT}{dr} \quad (A = 2\pi r L) \]
Problem 3

**KNOWN:**
Energy in, out, generation and depletion

**FIND:**
Temperature as function of time

**SCHEMATIC AND GIVEN DATA**

Air flow out

Outside surface

Air flow in at $T_a$

**STRATEGY:**
Lumped parameter analysis is mentioned, so we follow the derivation in class (In-Out+Gen=Change in Storage) except we cannot use the solution directly as there are more terms in the energy balance equation.

**ASSUMPTIONS:**

**SOLUTION:**

1. 
   $$\rho V c_p \frac{dT}{dt} = Q_a V - \dot{m} \lambda V(1000) + \rho a V c_p V (T_a - T) + h A (T_a - T)$$

2.
\[
\frac{Q_aV - m(\lambda V(1000) + (\frac{\rho_a c_p V_a V + h \Delta A)}{\rho V c_p})T_a}{\rho V c_p} = B
\]

\[
-\frac{\rho_a c_p V_a V - h \Delta A}{\rho V c_p} = C
\]

\[
\frac{dT}{dt} = B + CT
\]

\[
\int_{t_i}^{t} \frac{dT}{B + CT} = \int_{t_i}^{t} dt
\]

\[
\frac{1}{C} \ln \left( \frac{B + CT}{B + CT_i} \right) = t
\]

\[
T = \frac{B + CT_i}{C} \exp(Ct) - \frac{B}{C}
\]
**Problem 4**

**KNOWN:** Skin surface is in contact with a hot oven.

**FIND:** The depth to which skin is damaged by burns.

**SCHEMATIC AND GIVEN DATA**

1. Exposure time, \( t = 2 \) sec.
2. Oven temperature, \( T_2 = 200^\circ\)C.
3. Average skin temperature, \( T_1 = 33^\circ\)C.
4. Temperature at which skin is damaged, \( T_d = 62^\circ\)C.
5. Thermal diffusivity of skin, \( \alpha = 2.5 \times 10^{-7} \) m\(^2\)/s.

**STRATEGY:**
Following the solution chart (beginning of Chapter 4), we see that this problem is unsteady heat conduction in a semi-infinite geometry. Semi-infinite comes to mind since, for finite geometry, a shape and a size would be needed. Also, what may not be that obvious, the duration is somewhat short so only a layer near the surface will be heated. In a situation such as this, semi-infinite is an assumption and should be checked at the end of solution process.

**ASSUMPTIONS:**

1. The skin layer is thick compared to the damaged layer (semi-infinite region).
2. The temperature of the skin before touching the oven is uniform.

**SOLUTION:**

1.

\[
\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad T(x \to \infty, t) = 33^\circ\)C
\]
\[
T(x, t \leq 0) = 33^\circ\)C
\]
\[
T(x = 0, t) = 200^\circ\)C.
\]

For a semi-infinite sheet:
\[
\frac{T - T_i}{T_\infty - T_i} = 1 - \text{erf} \left[ \frac{x}{2\sqrt{\alpha t}} \right]
\]

\[
\frac{62 - 33}{200 - 33} = 1 - \text{erf} \left[ \frac{x}{2\sqrt{2.5 \times 10^{-7} (2)}} \right]
\]

\[
0.1737 = 1 - \text{erf} \left[ \frac{x}{1.414 \times 10^{-5}} \right]
\]

\[
0.8263 = \text{erf} \left[ \frac{x}{1.414 \times 10^{-5}} \right]
\]

For \( \text{erf} \phi = 0.8263, \phi = 0.96 \) (from Appendix B3)

Plugging in, \( x = 1.36 \text{mm} \)

2.

\[
q''_s = \frac{k (T_\infty - T_i)}{\sqrt{\pi \alpha t}} = \frac{0.43 \left[ \frac{W}{\text{m} \cdot \text{C}} \right] (200 - 33)[\text{C}]}{\sqrt{\pi \times 2.5 \times 10^{-7} \left[ \frac{m^2}{s} \right] \times 2 \text{ s}}} = 57296 \left[ \frac{W}{m^2} \right]
\]

3.

\[
E = \int_0^t q''_s \, dt = \int_0^t \frac{k(T_\infty - T_i)}{\sqrt{\pi \alpha}} \, t^{-\frac{1}{2}} \, dt
\]

\[
E = \frac{k(T_\infty - T_i)}{\frac{1}{2} \sqrt{\pi \alpha}} \, t^\frac{1}{2} = \frac{0.43 \left[ \frac{W}{\text{m} \cdot \text{C}} \right] (200 - 33)[\text{C}]}{\frac{1}{2} \sqrt{\pi \times 2.5 \times 10^{-7} \left[ \frac{m^2}{s} \right]}} \left( \sqrt{2 \left[ \text{s} \right]} \right) = 229184 \left[ \frac{J}{m^2} \right]
\]

4. \( x \propto \sqrt{t} \)
1. 2 BCs per each spatial variable.
2. Because bodies of cold blooded animals have a passive process of heat transfer and the steady state temperature would simply be a result of heat balance.
3. In a gas, the molecules are further apart. Molecular movements are most random and interactions are less frequent. Thus lowest thermal conductivity values are found in gases.
4. At very short times the propagating energy front does not know that the material is finite (i.e. it is not affected by 2nd BC). So it can be modeled as semi-infinite material.
5. 
   \[
   \frac{\alpha t}{L^2} = \frac{7 \times 10^{-4} \left[ \frac{m^2}{s} \right] \times 10 [s]}{(0.5 [m])^2} = 0.028 < 0.0625
   \]
   So, semi-infinite.

6. The cross section area \((2\pi rL)\) through which heat flows increases with radius \(r\). So, the temperature gradient \(dT/dr\) has to decrease with radius \(r\) to keep the heat flow constant at steady state.